

# ON A PROBLEM OF WEIGHTED LOW RANK APPROXIMATION OF MATRICES

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**Abstract.** We study a weighted low rank approximation that is inspired by a problem of constrained low rank approximation of matrices as initiated by the work of Golub, Hoffman, and Stewart (Linear Algebra and Its Applications, 88-89(1987), 317-327). Our results reduce to that of Golub, Hoffman, and Stewart in the limiting cases. We also propose an algorithm based on the alternating direction method and demonstrate convergence asserted in our theorems.

**Key words.** Weighted low rank approximation, singular value decomposition, alternating direction method

**AMS subject classifications.** 65F30, 65K05, 49M15, 49M30

**1. Introduction.** Let  $m$  and  $n$  be two natural numbers. For an integer  $r \leq \min\{m, n\}$  and a matrix  $A \in \mathbb{R}^{m \times n}$ , the standard low rank approximation problem can be formulated as

$$(1.1) \quad \min_{\substack{X \in \mathbb{R}^{m \times n} \\ r(X) \leq r}} \|A - X\|_F^2,$$

where  $r(X)$  denotes the rank of the matrix  $X$  and  $\|\cdot\|_F$  denotes the Frobenius norm of matrices.

It is well known that the solutions to this problem can be given using the singular value decompositions (SVDs) of  $A$  through the hard thresholding operations on the singular values: The solutions to (1.1) are given by

$$(1.2) \quad X^* = H_r(A) := U(A)\Sigma_r(A)V(A)^T,$$

where  $A = U(A)\Sigma(A)V(A)^T$  is a SVD of  $A$  and  $\Sigma_r(A)$  is the diagonal matrix obtained from  $\Sigma(A)$  by thresholding: keeping only  $r$  largest singular values and replacing other singular values by 0 along the diagonal. This is also referred to as Ektart-Young-Mirsky's theorem ([6]) and is closely related to the PCA method in statistics [3]. The solutions to (1.1) as given in (1.2) suffer from the fact that none of the entries of  $X$  is guaranteed to be preserved in  $X^*$ . In many real world problems, one has good reasons to keep certain entries of  $A$  unchanged while looking for a low rank approximation. For example, if SVD is used in quadrantally-symmetric two-dimensional (2-D) filter design, as pointed out in ([23, 24, 25]), it might lead to a degraded construction in some cases as it is not able to discriminate between the important and unimportant components of  $X$ . To address this problem, a weighted least squares matrix decomposition (WLR) method was first proposed by Shpak [25]. Following his idea of assigning different weights to discriminate between important and unimportant components of the test matrix, Lu, Pei, and Wang ([24]) designed a numerical procedure to find the best rank  $r$  approximation of the matrix  $A$  in the *weighted Frobenius* norm sense:

$$(1.3) \quad \min_{\substack{X \in \mathbb{R}^{m \times n} \\ r(X) \leq r}} \|(A - X) \odot W\|_F^2,$$

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where  $W \in \mathbb{R}^{m \times n}$  is a weight matrix and  $\odot$  denotes the Hadamard product.

In 1987, Golub, Hoffman, and Stewart were the first to consider the following *constrained* low rank approximation problem [1]:

Given  $A = (A_1 \ A_2) \in \mathbb{R}^{m \times n}$  with  $A_1 \in \mathbb{R}^{m \times k}$  and  $A_2 \in \mathbb{R}^{m \times (n-k)}$ , find  $\tilde{A}_2$  such that (with  $\tilde{A}_1 = A_1$ )

$$(1.4) \quad (\tilde{A}_1 \ \tilde{A}_2) = \arg \min_{\substack{X_1, X_2 \\ r(X_1 \ X_2) \leq r \\ X_1 = A_1}} \|(A_1 \ A_2) - (X_1 \ X_2)\|_F^2.$$

That is, Golub, Hoffman, and Stewart required that the first few columns,  $A_1$ , of  $A$  must be preserved when one looks for a low rank approximation of  $(A_1 \ A_2)$ . As in the standard low rank approximation, the constrained low-rank approximation problem of Golub, Hoffman, and Stewart also has a closed form solution.

**THEOREM 1.1.** [1] *With  $k = r(A_1)$  and  $r \geq k$ , the solutions  $\tilde{A}_2$  in (1.4) are given by*

$$(1.5) \quad \tilde{A}_2 = P_{A_1}(A_2) + H_{r-k}(P_{A_1}^\perp(A_2)),$$

where  $P_{A_1}$  and  $P_{A_1}^\perp$  are the projection operators to the column space of  $A_1$  and its orthogonal complement, respectively.

**REMARK 1.2.** According to Section 3 of ([1]), the matrix  $\tilde{A}_2$  is unique if and only if  $H_{r-k}(P_{A_1}^\perp(A_2))$  is unique, which means the  $(r-k)$ th singular value of  $P_{A_1}^\perp(A_2)$  is strictly greater than  $(r-k+1)$ th singular value. When  $\tilde{A}_2$  is not unique, the formula for  $\tilde{A}_2$  given in Theorem 1.1 should be understood as the membership of the set specified by the right-hand side of (1.5). We will use this convention in this paper.

Inspired by Theorem 1.1 above and motivated by applications in which  $A_1$  may contain noise, it makes more sense if we require  $\|X_1 - A_1\|_F$  small (as in the case of the total least squares) instead of asking for  $X_1 = A_1$ . This leads us to consider the following problem: Let  $\lambda > 0$  and  $W_\lambda = \begin{pmatrix} \lambda I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}$ , find  $(\hat{X}_1 \ \hat{X}_2)$  such that

$$(1.6) \quad (\hat{X}_1 \ \hat{X}_2) = \arg \min_{\substack{X_1, X_2 \\ r(X_1 \ X_2) \leq r}} \|((A_1 \ A_2) - (X_1 \ X_2)) W_\lambda\|_F^2.$$

This problem can be viewed as “approximately” preserving (controlled by a parameter  $\lambda$ ), instead of requiring exactly matching, in the first few columns. Indeed, as in a related work of background estimation, shadows and specularities removal from face images, and domain adaptation problems in computer vision and machine learning in ([2]), a closely related weighted low-rank approximation is shown to be more effective. Note that multiplying a matrix from right by  $W_\lambda$  is same as multiply  $\lambda$  to each element of the first  $k$  columns of that matrix and leaving the rest of the elements unchanged. As it turns out, this formulation can be viewed as generalized total least squares problem (GTLS) [18, 19]. Problem (1.6) is a special case of weighted low-rank approximation with a rank-one weight matrix and can be solved in closed form by using a single SVD of the given matrix  $(\lambda A_1 \ A_2)$  [18, 19]. A careful reader must also note that, both problems (1.4) and (1.6) can be cast as special cases of structured low-rank problems with element-wise weights [21, 26]. In this paper, we consider a more general point-wise multiplication with a matrix  $W = (W_1 \ W_2)$  of non-negative terms:

$$(1.7) \quad \min_{\substack{X_1, X_2 \\ r(X_1 \ X_2) \leq r}} \|((A_1 \ A_2) - (X_1 \ X_2)) \odot (W_1 \ W_2)\|_F^2.$$

This is the weighted low rank approximation problem studied first when  $W$  is an indicator weight for dealing with the missing data case ([10, 11]) and then for more general weight in machine learning, collaborative filtering, 2-D filter design, and computer vision [7, 12, 16, 23, 24, 25]. One can consider (1.7) as a special case of the weighted low-rank approximation problem defined in [23]:

$$(1.8) \quad \min_{X \in \mathbb{R}^{m \times n}} \|A - X\|_Q^2, \quad \text{subject to } r(X) \leq r,$$

where  $Q \in \mathbb{R}^{mn \times mn}$  is a symmetric positive definite weight matrix. Denote  $\|A - X\|_Q^2 := \text{vec}(A - X)^T Q \text{vec}(A - X)$ , where  $\text{vec}(\cdot)$  is an operator which maps the entries of  $\mathbb{R}^{m \times n}$  to  $\mathbb{R}^{mn \times 1}$ . Since  $Q$  is symmetric and positive definite, the problem (1.8) can be transformed into (1.7) through an unitary transformation. So, problem (1.7) and problem (1.8) are equivalent. Unlike problem (1.6) the weighted low rank approximation problem (1.7) has no closed form solution in general [7, 23]. Also note that the entry-wise multiplication is not associative with the regular matrix multiplication:  $(A \cdot B) \odot C \neq A \cdot (B \odot C)$ , and as a consequence, we lose the unitary invariance property in case of using the Frobenius norm. We are interested in finding out the limit behavior of the solutions to problem (1.7) when  $(W_1)_{ij} \rightarrow \infty$  and  $W_2 = \mathbb{1}$ , a matrix whose entries are equal to 1. One can expect that, with appropriate conditions, the solutions to (1.7) will converge and the limit is  $A_G$ . We will verify this with an estimate on the rate of convergence. We will also extend the convergence result to the unconstrained version of the problem (1.7) and propose a numerical algorithm to solve (1.7) for the special case of the weight matrix  $(W_1)_{ij} \rightarrow \infty$  and  $W_2 = \mathbb{1}$ .

The rest of the paper is organized as follows. In Section 2, we state our main results. Their proofs are given in Section 3. In Section 4, we present the numerical solution to problem (1.7) for the special case of the weight and discuss the convergence of our algorithm. Numerical results verifying our main results are given in Section 5.

**2. Main results.** Let  $(\tilde{X}_1(W), \tilde{X}_2(W))$  be a solution to (1.7). Denote  $\mathcal{A} = P_{A_1}^\perp(A_2)$  and  $\tilde{\mathcal{A}} = P_{\tilde{X}_1(W)}^\perp(A_2)$ . Also denote  $s = r(\mathcal{A})$  and let the ordered non-zero singular values of  $\mathcal{A}$  be  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_s > 0$ . Let  $\lambda_j = \min_{1 \leq i \leq m} (W_1)_{ij}$  and  $\lambda = \min_{1 \leq j \leq k} \lambda_j$ .

**THEOREM 2.1.** *Let  $W_2 = \mathbb{1}_{m \times (n-k)}$ . If  $\sigma_{r-k} > \sigma_{r-k+1}$ , then*

$$(\tilde{X}_1(W), \tilde{X}_2(W)) = A_G + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty,$$

where  $A_G = (A_1, \tilde{A}_2)$  is defined to be the unique solution to (1.4).

**REMARK 2.2.** (i). The assertion of the uniqueness of  $A_G$  is due to the assumption  $\sigma_{r-k} > \sigma_{r-k+1}$  (see the Remark 1.2). (ii). As in ([21]), with proper condition one can find  $(\tilde{X}_1(W), \tilde{X}_2(W)) \rightarrow A_G$  as  $(W_1)_{ij} \rightarrow \infty$  and  $W_2 = \mathbb{1}$ . We should mention, however, it does not give the convergence rate as proposed in Theorem 2.1.

Next, if we do not know  $r$  but still want to reduce the rank in our approximation, consider the unconstrained version of (1.7): for  $\tau > 0$ ,

$$(2.1) \quad \min_{X_1, X_2} \left\{ \|((A_1, A_2) - (X_1, X_2)) \odot (W_1, W_2)\|_F^2 + \tau r(X_1, X_2) \right\}.$$

Again one can expect that the solutions to (2.1) will converge to  $A_G$  as  $(W_1)_{ij} \rightarrow \infty$  and  $(W_2)_{ij} \rightarrow 1$ . Define  $\mathcal{A}_G^r$ ,  $0 \leq r \leq \min\{m, n\}$ , to be the set of all solutions to (1.4).

Let  $(\hat{X}_1(W) \ \hat{X}_2(W))$  be a solution to (2.1).

**THEOREM 2.3.** *Every accumulation point of  $(\hat{X}_1(W) \ \hat{X}_2(W))$  as  $(W_1)_{ij} \rightarrow \infty, (W_2)_{ij} \rightarrow 1$  belongs to  $\bigcup_{0 \leq r \leq \min\{m, n\}} \mathcal{A}_G^r$ .*

**THEOREM 2.4.** *Assume that  $\sigma_1 > \sigma_2 > \dots > \sigma_s > 0$ . Denote  $\sigma_0 := \infty$  and  $\sigma_{s+1} := 0$ . Then the accumulation point of the sequence  $(\hat{X}_1(W) \ \hat{X}_2(W))$ , as  $(W_1)_{ij} \rightarrow \infty$  and  $(W_2)_{ij} \rightarrow 1$  is unique; and this unique accumulation point is given by*

$$(A_1 \ P_{A_1}(A_2) + H_{r^*}(P_{A_1}^\perp(A_2)))$$

with  $r^*$  satisfying

$$\sigma_{r^*+1}^2 \leq \tau < \sigma_{r^*}^2.$$

**REMARK 2.5.** For the case when  $P_{A_1}^\perp(A_2)$  has repeated singular values, we leave it to the reader to verify the following more general statement by using a similar argument: Let  $\hat{\sigma}_1 > \hat{\sigma}_2 > \dots > \hat{\sigma}_t > 0$  be the singular values of  $P_{A_1}^\perp(A_2)$  with multiplicity  $k_1, k_2, \dots, k_t$  respectively. Note that  $\sum_{i=1}^t k_i = s$ . Let  $\sigma_{p^*+1}^2 \leq \tau < \sigma_{p^*}^2$ , where  $\sigma_{p^*}$  has multiplicity  $k_{p^*}$ . Then the accumulation points of the set  $(\hat{X}_1(W), \hat{X}_2(W))$ , as  $(W_1)_{ij} \rightarrow \infty, (W_2)_{ij} \rightarrow 1$ , belongs to the set  $\bigcup_{r^*} \mathcal{A}_G^{r^*}$ , where  $1 + \sum_{i=1}^{p^*-1} k_i \leq r^* < \sum_{i=1}^{p^*} k_i$ .

**3. Proofs.** To prove Theorem 2.1, we first establish the following lemmas.

**LEMMA 3.1.** *As  $(W_1)_{ij} \rightarrow \infty$  and  $W_2 = \mathbb{1}$ , we have the following estimates.*

- (i)  $\tilde{X}_1(W) = A_1 + O(\frac{1}{\lambda})$ .
- (ii)  $P_{\tilde{X}_1(W)}(A_2) = P_{A_1}(A_2) + O(\frac{1}{\lambda})$ .
- (iii)  $P_{\tilde{X}_1(W)}^\perp(A_2) = P_{A_1}^\perp(A_2) + O(\frac{1}{\lambda})$ .

*Proof.* (i). Note that,

$$\begin{aligned} & \| (A_1 - \tilde{X}_1(W)) \odot W_1 \|_F^2 + \| A_2 - \tilde{X}_2(W) \|_F^2 \\ &= \min_{\substack{X_1, X_2 \\ r(X_1 \ X_2) \leq r}} (\| (A_1 - X_1) \odot W_1 \|_F^2 + \| A_2 - X_2 \|_F^2) \\ &\leq \| A_2 \|_F^2 \text{ (by taking } (X_1 \ X_2) = (A_1 \ 0)) \\ &= m_1 \text{ (say).} \end{aligned}$$

Then  $\sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq k}} ((A_1)_{ij} - (\tilde{X}_1(W))_{ij})^2 (W_1)_{ij}^2 \leq m_1$  and so

$$|(A_1)_{ij} - (\tilde{X}_1(W))_{ij}| \leq \frac{\sqrt{m_1}}{(W_1)_{ij}}; \quad 1 \leq i \leq m, 1 \leq j \leq k.$$

Thus

$$\tilde{X}_1(W) = A_1 + O(\frac{1}{\lambda}) \text{ as } \lambda \rightarrow \infty.$$

(ii). For simplicity, let us assume  $\text{r}(A_1) = k$ , full rank. If  $\text{r}(A_1) = l < k$ , then  $A_1$  can be replaced by a matrix with  $l$  linearly independent columns chosen from  $A_1$  [1]. We use the  $QR$  decomposition of  $A = (A_1 \ A_2)$ . Let

$$(A_1 \ A_2) = QR = (Q_1 \ Q_2 \ Q_3) \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & 0 \end{pmatrix},$$

where  $Q \in \mathbb{R}^{m \times m}$  is an orthogonal matrix with block matrices  $Q_1$ ,  $Q_2$ , and  $Q_3$  of sizes  $m \times k$ ,  $m \times (n - k)$ , and  $m \times (m - n)$ , respectively, and the matrices  $R_{11}$  and  $R_{22}$  are both upper triangular. Therefore,

$$(3.1) \quad \begin{cases} A_1 = Q_1 R_{11}, \\ A_2 = Q_1 R_{12} + Q_2 R_{22}. \end{cases}$$

Note that  $Q_1 R_{12} = P_{A_1}(A_2)$  and  $Q_2 R_{22} = P_{A_1}^\perp(A_2)$ . By (i), we see that  $\text{r}(\tilde{X}_1(W)) = k$ , for all large  $(W_1)_{ij}$ . We now look at the  $QR$  decomposition of  $\tilde{X}_1(W)$ :

$$(3.2) \quad \tilde{X}_1(W) = Q_1(W) R_{11}(W),$$

where  $Q_1(W)$  is column orthogonal ( $Q_1^T(W) Q_1(W) = I_k$ ), and  $R_{11}(W)$  is upper triangular. The  $QR$  decomposition can be obtained via the Gram-Schmidt process. If we write the matrices as collection of column vectors:

$$\tilde{X}_1(W) = (x_1(W) \ x_2(W) \ \cdots \ x_k(W)), \quad Q_1(W) = (q_1(W) \ q_2(W) \ \cdots \ q_k(W)),$$

and

$$A_1 = (a_1 \ a_2 \ \cdots \ a_k), \quad Q_1 = (q_1 \ q_2 \ \cdots \ q_k),$$

where  $x_i(W), q_i(W), a_i, q_i \in \mathbb{R}^m$ ,  $i = 1, 2, \dots, k$ , then by (i),

$$(3.3) \quad x_i(W) = a_i + O\left(\frac{1}{\lambda_i}\right), \quad \lambda_i \rightarrow \infty.$$

So, as  $\lambda_1 \rightarrow \infty$ ,

$$q_1(W) = \frac{x_1(W)}{\|x_1(W)\|_2} = \frac{a_1 + O(\frac{1}{\lambda_1})}{\|a_1 + O(\frac{1}{\lambda_1})\|_2} = \frac{a_1}{\|a_1\|_2} + O\left(\frac{1}{\lambda_1}\right) = q_1 + O\left(\frac{1}{\lambda_1}\right),$$

where  $\|\cdot\|_2$  denotes the  $\ell_2$  norm of vectors. Similarly, we see that

$$\langle x_2(W), q_1(W) \rangle = \langle a_2, q_1 \rangle + O\left(\frac{1}{\min\{\lambda_1, \lambda_2\}}\right), \quad \min\{\lambda_1, \lambda_2\} \rightarrow \infty,$$

and

$$\begin{aligned} & x_2(W) - \langle x_2(W), q_1(W) \rangle q_1(W) \\ &= a_2 - \langle a_2, q_1 \rangle q_1 + O\left(\frac{1}{\min\{\lambda_1, \lambda_2\}}\right), \quad \min\{\lambda_1, \lambda_2\} \rightarrow \infty, \end{aligned}$$

which leads to

$$\begin{aligned} q_2(W) &= \frac{x_2(W) - \langle x_2(W), q_1(W) \rangle q_1(W)}{\|x_2(W) - \langle x_2(W), q_1(W) \rangle q_1(W)\|_2} \\ &= q_2 + O\left(\frac{1}{\min\{\lambda_1, \lambda_2\}}\right), \quad \min\{\lambda_1, \lambda_2\} \rightarrow \infty. \end{aligned}$$

Continuing this process we obtain, as  $\lambda \rightarrow \infty$ ,

$$Q_1(W) = (q_1 \ q_2 \cdots q_k) + O\left(\frac{1}{\min\{\lambda_1, \dots, \lambda_k\}}\right) = Q_1 + O\left(\frac{1}{\lambda}\right).$$

Finally, we have

$$\begin{aligned} P_{\tilde{X}_1(W)}(A_2) &= Q_1(W)Q_1(W)^T A_2 \\ &= \left(Q_1 + O\left(\frac{1}{\lambda}\right)\right) \left(Q_1 + O\left(\frac{1}{\lambda}\right)\right)^T A_2 \\ &= P_{A_1}(A_2) + O\left(\frac{1}{\lambda}\right), \end{aligned}$$

as  $\lambda \rightarrow \infty$ .

(iii) We know that

$$P_{\tilde{X}_1(W)}(A_2) + P_{\tilde{X}_1(W)}^\perp(A_2) = A_2 = P_{A_1}(A_2) + P_{A_1}^\perp(A_2).$$

Using (ii)

$$P_{A_1}(A_2) + O\left(\frac{1}{\lambda}\right) + P_{\tilde{X}_1(W)}^\perp(A_2) = P_{A_1}(A_2) + P_{A_1}^\perp(A_2), \quad \lambda \rightarrow \infty.$$

Therefore,

$$(3.4) \quad P_{\tilde{X}_1(W)}^\perp(A_2) = P_{A_1}^\perp(A_2) + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty.$$

This completes the proof of Lemma 3.1.  $\square$

REMARK 3.2. For the case when there is an uniform weight in  $(W_1)_{ij} = \lambda > 0$ , one might refer to [22] for an alternative proof of Lemma 3.1. But the proof in [22] can not be applied in the more general case as in Lemma 3.1.

LEMMA 3.3. *If  $\sigma_{r-k} > \sigma_{r-k+1}$ , then*

$$(3.5) \quad H_{r-k}(\tilde{\mathcal{A}}) = H_{r-k}(\mathcal{A}) + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty.$$

*Proof.* Let the SVDs of  $\mathcal{A}$ ,  $\tilde{\mathcal{A}}$  be given by

$$(3.6) \quad \mathcal{A} = U\Sigma V^T = (U_1 \ U_2) \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} =: \mathcal{A}_1 + \mathcal{A}_2,$$

$$(3.7) \quad \tilde{\mathcal{A}} = \tilde{U}\tilde{\Sigma}\tilde{V}^T = (\tilde{U}_1 \ \tilde{U}_2) \begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & \tilde{\Sigma}_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^T \\ \tilde{V}_2^T \end{pmatrix} =: \tilde{\mathcal{A}}_1 + \tilde{\mathcal{A}}_2,$$

such that  $U, \tilde{U} \in \mathbb{R}^{m \times m}$ ,  $V, \tilde{V} \in \mathbb{R}^{(n-k) \times (n-k)}$ , and  $\Sigma, \tilde{\Sigma} \in \mathbb{R}^{m \times (n-k)}$  with  $\Sigma$  and  $\tilde{\Sigma}$  being diagonal matrices containing singular values of  $\mathcal{A}$  and  $\tilde{\mathcal{A}}$ , respectively, arranged

in a non-increasing order;  $U_1, \tilde{U}_1 \in \mathbb{R}^{m \times (r-k)}, U_2, \tilde{U}_2 \in \mathbb{R}^{m \times (m-r+k)}, V_1, \tilde{V}_1 \in \mathbb{R}^{(n-k) \times (r-k)}$ , and  $V_2, \tilde{V}_2 \in \mathbb{R}^{(n-k) \times (n-r)}$ . Using (3.6) and (3.7) we have

$$(3.8) \quad \tilde{\mathcal{A}} = \tilde{\mathcal{A}}_1 + \tilde{\mathcal{A}}_2 = \mathcal{A}_1 + \mathcal{A}_2 + E = \mathcal{A} + E.$$

Then by (iii) of Lemma 3.1, we know that  $E = O(\frac{1}{\lambda})$ ,  $\lambda \rightarrow \infty$ . We will need the following result.

LEMMA 3.4. [8] *Let  $\tilde{\mathcal{A}} = \mathcal{A} + E$  and  $\sigma \neq 0$  be a non-repeating singular value of the matrix  $\mathcal{A}$  with  $u$  and  $v$  being left and right singular vectors respectively. Then as  $\lambda \rightarrow \infty$ , there is a unique singular value  $\tilde{\sigma}$  of  $\tilde{\mathcal{A}}$  such that*

$$(3.9) \quad \tilde{\sigma} = \sigma + u^T E v + O(\|E\|^2).$$

The lemma above will allow us to estimate  $\Sigma - \tilde{\Sigma}$ . Indeed, with the non-increasing arrangement of the singular values in  $\Sigma$  and  $\tilde{\Sigma}$ , and the fact that  $E = O(\frac{1}{\lambda})$  as  $\lambda \rightarrow \infty$ , Lemma 3.4 immediately implies that

$$(3.10) \quad \Sigma_1 - \tilde{\Sigma}_1 = O(\frac{1}{\lambda}) \quad \text{and} \quad \Sigma_2 - \tilde{\Sigma}_2 = O(\frac{1}{\lambda}) \quad \text{as } \lambda \rightarrow \infty.$$

We pause on the proof of Lemma 3.3 further and introduce some useful tools. Our goal is to compare  $\mathcal{A}_1$  and  $\tilde{\mathcal{A}}_1$ . This leads us to consider the column spaces of  $\mathcal{A}_1$  and  $\tilde{\mathcal{A}}_1$ . One way to measure the distance between two subspaces is to measure the angle between them [9]. Davis and Kahan measured the difference of the angles between the invariant subspaces of a Hermitian matrix and its perturbed form as a function of their perturbation and the separation of their spectra. Wedin proposed a more generalized form. Using the generalized  $\sin \theta$  Theorem of Wedin ([5]), the following results can be achieved (see Section 4.4 in [5]).

LEMMA 3.5. [5] *Let  $\mathcal{A}$  and  $\tilde{\mathcal{A}}$  be as given in (3.8). Assume there exists an  $\alpha \geq 0$  and a  $\delta > 0$  such that*

$$\sigma_{\min}(\tilde{\mathcal{A}}_1) \geq \alpha + \delta \quad \text{and} \quad \sigma_{\max}(\mathcal{A}_2) \leq \alpha,$$

then

$$(3.11) \quad \|\mathcal{A}_1 - \tilde{\mathcal{A}}_1\| \leq \|E\| \left( 3 + \frac{\|\mathcal{A}_2\|}{\delta} + \frac{\|\tilde{\mathcal{A}}_2\|}{\delta} \right).$$

*Proof of Lemma 3.3 continued.* Note that  $r(\mathcal{A}_1) = r(\tilde{\mathcal{A}}_1) = r - k$ , and, since  $\sigma_{r-k} > \sigma_{r-k+1}$ , we can choose  $\delta$  such that

$$\delta \geq \frac{1}{2}(\sigma_{r-k} - \sigma_{r-k+1}) > 0.$$

Note that, in this way, for all large  $\lambda$  the assumption of Lemma 3.5 will be satisfied. Since  $\mathcal{A}_1 = H_{r-k}(\mathcal{A})$  and  $\tilde{\mathcal{A}}_1 = H_{r-k}(\tilde{\mathcal{A}})$ , (3.11) can be written as

$$(3.12) \quad \|H_{r-k}(\mathcal{A}) - H_{r-k}(\tilde{\mathcal{A}})\| \leq \|E\| \left( 3 + \frac{\|\mathcal{A}_2\|}{\delta} + \frac{\|\tilde{\mathcal{A}}_2\|}{\delta} \right).$$

Since  $\mathcal{A}_2$  is fixed,  $\|\mathcal{A}_2\| = O(1)$  as  $\lambda \rightarrow \infty$ . On the other hand, by (3.10), as  $\lambda \rightarrow \infty$ ,

$$\tilde{\mathcal{A}}_2 = \tilde{U}_2 \tilde{\Sigma}_2 \tilde{V}_2^T = \tilde{U}_2 (\Sigma_2 + O(\frac{1}{\lambda})) \tilde{V}_2^T = \tilde{U}_2 \Sigma_2 \tilde{V}_2^T + O(\frac{1}{\lambda} \tilde{U}_2 \tilde{V}_2^T).$$

Now the unitary invariance of the matrix norm implies,

$$\|\tilde{\mathcal{A}}_2\| \leq \|\tilde{U}_2 \Sigma_2 \tilde{V}_2^T\| + O\left(\frac{1}{\lambda} \|\tilde{U}_2 \tilde{V}_2^T\|\right) = \|\Sigma_2\| + O\left(\frac{1}{\lambda}\right),$$

which is bounded as  $\lambda \rightarrow \infty$ . Therefore (3.12) becomes

$$(3.13) \quad \|H_{r-k}(\mathcal{A}) - H_{r-k}(\tilde{\mathcal{A}})\| \leq C\|E\|,$$

for some constant  $C > 0$  and for all large  $\lambda \rightarrow \infty$ . Thus

$$H_{r-k}(\tilde{\mathcal{A}}) = H_{r-k}(\mathcal{A}) + O\left(\frac{1}{\lambda}\right), \lambda \rightarrow \infty,$$

since  $E = O\left(\frac{1}{\lambda}\right)$  as  $\lambda \rightarrow \infty$ . This completes the proof of Lemma 3.3.  $\square$

*Proof of Theorem 2.1.* The proof is a consequence of Lemmas 3.1 and 3.3.  $\square$

*Proof of Theorem 2.3.* Let  $\hat{X}(W) = (\hat{X}_1(W) \ \hat{X}_2(W))$ . We need to verify that  $\{\hat{X}(W)\}_W$  is a bounded set and every accumulation point is a solution to (1.4) for some  $r$ . Since  $(\hat{X}_1(W) \ \hat{X}_2(W))$  is a solution to (2.1), we have

$$(3.14) \quad \begin{aligned} & \|(A_1 - \hat{X}_1(W)) \odot W_1\|_F^2 + \|(A_2 - \hat{X}_2(W)) \odot W_2\|_F^2 + \tau r(\hat{X}_1(W) \ \hat{X}_2(W)) \\ & \leq \|(A_1 - X_1) \odot W_1\|_F^2 + \|(A_2 - X_2) \odot W_2\|_F^2 + \tau r(X_1 \ X_2). \end{aligned}$$

for all  $(X_1 \ X_2)$ . By choosing  $X_1 = A_1, X_2 = 0$ , we can obtain a constant  $m_3 := \|A_2 \odot W_2\|_F^2 + \tau r(A_1 \ 0)$  such that  $\|(A_1 - \hat{X}_1(W)) \odot W_1\|_F^2 + \|(A_2 - \hat{X}_2(W)) \odot W_2\|_F^2 \leq m_3$ . Therefore,  $\{\hat{X}_1(W) \ \hat{X}_2(W)\}$  is bounded. Let  $(X_1^{**} \ X_2^{**})$  be an accumulation point of the sequence. We only need to show that  $(X_1^{**} \ X_2^{**}) \in \bigcup_r \mathcal{A}_G^r$ . As in the proof of Lemma 3.1 (i), we can show that

$$(3.15) \quad \lim_{\substack{(W_1)_{ij} \rightarrow \infty \\ (W_2)_{ij} \rightarrow 1}} \hat{X}_1(W) = A_1.$$

Now, taking limit and setting  $X_1 = A_1$  in (3.14), we can obtain,

$$(3.16) \quad \|A_2 - X_2^{**}\|_F^2 + \tau r(A_1 \ X_2^{**}) \leq \|A_2 - X_2\|_F^2 + \tau r(A_1 \ X_2),$$

for all  $X_2$ . If we denote  $r^{**} = r(A_1 \ X_2^{**})$ , then for  $X_2$  with  $r(A_1 \ X_2) \leq r^{**}$ , (3.16) yields

$$(3.17) \quad \|A_2 - X_2^{**}\|_F^2 \leq \|A_2 - X_2\|_F^2.$$

So,  $X_2^{**}$  is a solution to the problem of Golub, Hoffman, and Stewart. Thus, by Theorem 1.1,

$$X_2^{**} = P_{A_1}(A_2) + H_{r^{**}-k}(P_{A_1}^\perp(A_2)).$$

This, together with (3.15) completes the proof.  $\square$

*Proof of Theorem 2.4.* Let  $\hat{X}(W) = (\hat{X}_1(W) \ \hat{X}_2(W))$  solve the minimization problem (2.1). For convenience, we will drop the dependence on  $W$  in our notations. Then  $\hat{X}$  satisfies

$$(3.18) \quad \begin{aligned} & \|(A_1 - \hat{X}_1) \odot W_1\|_F^2 + \|(A_2 - \hat{X}_2) \odot W_2\|_F^2 + \tau r(\hat{X}_1 \ \hat{X}_2) \\ & \leq \|(A_1 - X_1^\dagger) \odot W_1\|_F^2 + \|(A_2 - X_2^\dagger) \odot W_2\|_F^2 + \tau r(\hat{X}_1 \ \hat{X}_2), \end{aligned}$$



for all  $X^\dagger = (X_1^\dagger \ X_2^\dagger) \in \mathbb{R}^{m \times n}$ . By choosing  $X_1^\dagger = A_1$  and  $X_1^\dagger = \hat{X}_2$  in (3.18) we obtain

$$\sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq k}} ((A_1)_{ij} - (\hat{X}_1)_{ij})^2 (W_1)_{ij}^2 \leq \tau r(A_1 \ \hat{X}_2) - \tau r(\hat{X}_1 \ \hat{X}_2) =: C.$$

Therefore,

$$(3.19) \quad \hat{X}_1 \rightarrow A_1, \quad (W_1)_{ij} \rightarrow \infty.$$

Next we choose  $X_1^\dagger = \hat{X}_1$  in (3.18) and find, for all  $X_2^\dagger$ ,

$$(3.20) \quad \|(A_2 - \hat{X}_2) \odot W_2\|_F^2 + \tau r(\hat{X}_1 \ \hat{X}_2) \leq \|(A_2 - X_2^\dagger) \odot W_2\|_F^2 + \tau r(\hat{X}_1 \ X_2^\dagger).$$

As in the proof of (ii) of Lemma 3.1, assume  $r(A_1) = k$  and consider a  $QR$  decomposition of  $A$ :

$$A = QR = Q(R_1 \ R_2) = Q \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & 0 \end{pmatrix}.$$

Write  $\hat{R} := Q^T \hat{X} = (\hat{R}_1 \ \hat{R}_2) = \begin{pmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \\ \hat{R}_{31} & \hat{R}_{32} \end{pmatrix}$  and let  $R^\dagger := (R_1^\dagger \ R_2^\dagger) = \begin{pmatrix} R_{11}^\dagger & R_{12}^\dagger \\ R_{21}^\dagger & R_{22}^\dagger \\ R_{31}^\dagger & R_{32}^\dagger \end{pmatrix}$

be in compatible block partitions. Since the rank of a matrix is invariant under an unitary transformation, (3.20) can be rewritten as

$$(3.21) \quad \begin{aligned} & \|(A_2 - \hat{X}_2) \odot W_2\|_F^2 + \tau r(Q^T \hat{X}_1 \ Q^T \hat{X}_2) \\ & \leq \|(A_2 - X_2^\dagger) \odot W_2\|_F^2 + \tau r(Q^T \hat{X}_1 \ Q^T X_2^\dagger). \end{aligned}$$

When  $\lambda$  is large enough,  $\hat{R}_{11}$  is nonsingular by (3.19) and the fact that  $r(A_1) = k$  and we can perform the row and column operations on the second term on left hand side of (3.21) to get:

$$\|(A_2 - \hat{X}_2) \odot W_2\|_F^2 + \tau r \begin{pmatrix} \hat{R}_{11} & 0 \\ 0 & \hat{R}_{22} - \hat{R}_{21} \hat{R}_{11}^{-1} \hat{R}_{12} \\ 0 & \hat{R}_{32} - \hat{R}_{31} \hat{R}_{11}^{-1} \hat{R}_{12} \end{pmatrix},$$

which is equal to

$$\|(A_2 - \hat{X}_2) \odot W_2\|_F^2 + \tau k + \tau r \begin{pmatrix} \hat{R}_{22} - \hat{R}_{21} \hat{R}_{11}^{-1} \hat{R}_{12} \\ \hat{R}_{32} - \hat{R}_{31} \hat{R}_{11}^{-1} \hat{R}_{12} \end{pmatrix}.$$

Performing the similar operations on the right hand side we obtain

$$\|(A_2 - X_2^\dagger) \odot W_2\|_F^2 + \tau r(\hat{R}_{11}) + \tau r \begin{pmatrix} R_{22}^\dagger - \hat{R}_{21} \hat{R}_{11}^{-1} R_{12}^\dagger \\ R_{32}^\dagger - \hat{R}_{31} \hat{R}_{11}^{-1} R_{12}^\dagger \end{pmatrix}.$$

Substituting these back in (3.21) we obtain

$$(3.22) \quad \begin{aligned} & \|(A_2 - \hat{X}_2) \odot W_2\|_F^2 + \tau r \begin{pmatrix} \hat{R}_{22} - \hat{R}_{21} \hat{R}_{11}^{-1} \hat{R}_{12} \\ \hat{R}_{32} - \hat{R}_{31} \hat{R}_{11}^{-1} \hat{R}_{12} \end{pmatrix} \\ & \leq \|(A_2 - X_2^\dagger) \odot W_2\|_F^2 + \tau r \begin{pmatrix} R_{22}^\dagger - \hat{R}_{21} \hat{R}_{11}^{-1} R_{12}^\dagger \\ R_{32}^\dagger - \hat{R}_{31} \hat{R}_{11}^{-1} R_{12}^\dagger \end{pmatrix}, \end{aligned}$$

for all  $R_{12}^\dagger, R_{22}^\dagger$ , and  $R_{32}^\dagger$ . From Theorem 2.3, we know that  $(\hat{R}_1 \ \hat{R}_2)$  has accumulation points which belong to  $\bigcup_{0 \leq r \leq \min\{m,n\}} \mathcal{A}_G^r$ . We are going to show that

$\lim_{\substack{(W_1)_{ij} \rightarrow \infty \\ (W_2)_{ij} \rightarrow 1}} \hat{R}_2$  indeed exists. Assume  $\lim_{\substack{(W_1)_{ij} \rightarrow \infty \\ (W_2)_{ij} \rightarrow 1}} \begin{pmatrix} \hat{R}_{12} \\ \hat{R}_{22} \\ \hat{R}_{32} \end{pmatrix} = \begin{pmatrix} \hat{R}_{12}^* \\ \hat{R}_{22}^* \\ \hat{R}_{32}^* \end{pmatrix}$  be an accumulation point. From (3.19), using the fact that  $\hat{R}_{11} \rightarrow R_{11}, \hat{R}_{21} \rightarrow 0$ , and  $\hat{R}_{31} \rightarrow 0$ , as  $(W_1)_{ij} \rightarrow \infty, (W_2)_{ij} \rightarrow 1$  in (3.22) we get

$$(3.23) \quad \|A_2 - \hat{X}_2^*\|_F^2 + \tau r \begin{pmatrix} \hat{R}_{22}^* \\ \hat{R}_{32}^* \end{pmatrix} \leq \|A_2 - X_2^\dagger\|_F^2 + \tau r \begin{pmatrix} R_{22}^\dagger \\ R_{32}^\dagger \end{pmatrix},$$

for all  $R_{12}^\dagger, R_{22}^\dagger$ , and  $R_{32}^\dagger$ . Since Frobenius norm is unitarily invariant, (3.23) reduces to

$$(3.24) \quad \left\| \begin{pmatrix} R_{12} \\ R_{22} \\ 0 \end{pmatrix} - \begin{pmatrix} \hat{R}_{12}^* \\ \hat{R}_{22}^* \\ \hat{R}_{32}^* \end{pmatrix} \right\|_F^2 + \tau r \begin{pmatrix} \hat{R}_{22}^* \\ \hat{R}_{32}^* \end{pmatrix} \leq \left\| \begin{pmatrix} R_{12} \\ R_{22} \\ 0 \end{pmatrix} - \begin{pmatrix} R_{12}^\dagger \\ R_{22}^\dagger \\ R_{32}^\dagger \end{pmatrix} \right\|_F^2 + \tau r \begin{pmatrix} R_{22}^\dagger \\ R_{32}^\dagger \end{pmatrix},$$

for all  $R_{12}^\dagger, R_{22}^\dagger$ , and  $R_{32}^\dagger$ . Substituting  $R_{22}^\dagger = \hat{R}_{22}^*, R_{32}^\dagger = \hat{R}_{32}^*$ , and  $R_{12}^\dagger = R_{12}$ , in (3.24) yields

$$\|R_{12} - \hat{R}_{12}^*\|_F^2 \leq 0,$$

which implies  $\lim_{\substack{(W_1)_{ij} \rightarrow \infty \\ (W_2)_{ij} \rightarrow 1}} \hat{R}_{12} = R_{12}$ . Next, substituting  $R_{12}^\dagger = \hat{R}_{12}^*$  in (3.24) we find

$$(3.25) \quad \left\| \begin{pmatrix} R_{22} \\ 0 \end{pmatrix} - \begin{pmatrix} \hat{R}_{22}^* \\ \hat{R}_{32}^* \end{pmatrix} \right\|_F^2 + \tau r \begin{pmatrix} \hat{R}_{22}^* \\ \hat{R}_{32}^* \end{pmatrix} \leq \left\| \begin{pmatrix} R_{22} \\ 0 \end{pmatrix} - \begin{pmatrix} R_{22}^\dagger \\ R_{32}^\dagger \end{pmatrix} \right\|_F^2 + \tau r \begin{pmatrix} R_{22}^\dagger \\ R_{32}^\dagger \end{pmatrix},$$

for all  $R_{22}^\dagger, R_{32}^\dagger$ . Let  $\bar{R}^* = \begin{pmatrix} \hat{R}_{22}^* \\ \hat{R}_{32}^* \end{pmatrix}$  and  $r^* = r(\bar{R}^*)$ , then (3.25) implies

$$(3.26) \quad \left\| \begin{pmatrix} R_{22} \\ 0 \end{pmatrix} - \bar{R}^* \right\|_F^2 \leq \left\| \begin{pmatrix} R_{22} \\ 0 \end{pmatrix} - R^* \right\|_F^2,$$

for all  $R^* \in \mathbb{R}^{(m-k) \times (n-k)}$  with  $r(R^*) \leq r^*$ . So  $\bar{R}^*$  solves a problem of classical low-rank approximation of  $\begin{pmatrix} R_{22} \\ 0 \end{pmatrix}$ . Note that,  $Q_2 \begin{pmatrix} R_{22} \\ 0 \end{pmatrix} = P_{A_1}^\perp(A_2)$  (see (3.1)) and it is assumed that  $P_{A_1}^\perp(A_2)$  has distinct singular values. So there exists a unique  $\bar{R}^*$  which is given by  $\bar{R}^* = H_{r^*} \begin{pmatrix} R_{22} \\ 0 \end{pmatrix}$  as in (1.2)). Therefore there is only one accumulation point of  $\{\hat{R}_2\}$  and so  $\lim_{\substack{(W_1)_{ij} \rightarrow \infty \\ (W_2)_{ij} \rightarrow 1}} \hat{R}_2$  exists. It remains for us to identify this unique accumulation point. Assume that

$$\begin{pmatrix} R_{22} \\ 0 \end{pmatrix} = Q^T \Sigma P$$

is a SVD of  $\begin{pmatrix} R_{22} \\ 0 \end{pmatrix}$ . Then, for any  $R^* \in \mathbb{R}^{(m-k) \times (n-k)}$ , (3.25) gives

$$(3.27) \quad \begin{aligned} & \|\Sigma - Q\bar{R}^*P^T\|_F^2 + \tau \text{r}(Q\bar{R}^*P^T) \\ & \leq \|\Sigma - QR^*P^T\|_F^2 + \tau \text{r}(QR^*P^T), \end{aligned}$$

Since  $r^* = \text{r}(\bar{R}^*)$  and  $Q\bar{R}^*P^T = \text{diag}(\sigma_1 \sigma_2 \cdots \sigma_{r^*} 0 \cdots 0)$ , choosing  $R^*$  such that

$$QR^*P^T = \text{diag}(\sigma_1 \sigma_2 \cdots \sigma_{r^*+1} 0 \cdots 0),$$

and using (3.27) we find

$$\sigma_{r^*+2}^2 + \cdots + \sigma_n^2 + \tau \geq \sigma_{r^*+1}^2 + \sigma_{r^*+2}^2 + \cdots + \sigma_n^2.$$

Next we choose  $R^*$  such that

$$QR^*P^T = \text{diag}(\sigma_1 \sigma_2 \cdots \sigma_{r^*-1} 0 \cdots 0),$$

and so  $\text{r}(R^*) = r^* - 1 < r^*$ . Now (3.26) and Ektart-Young-Mirsky's theorem then imply the equality in (3.27) can not hold. So,

$$\sigma_{r^*}^2 + \cdots + \sigma_n^2 - \tau > \sigma_{r^*+1}^2 + \sigma_{r^*+2}^2 + \cdots + \sigma_n^2.$$

Therefore, we obtain

$$(3.28) \quad \sigma_{r^*}^2 > \tau \geq \sigma_{r^*+1}^2.$$

It is easy to see that if (3.28) holds then  $\text{r}(\bar{R}^*) = r^*$ . So,

$$\text{r}(\bar{R}^*) = r^* \text{ if and only if } \sigma_{r^*}^2 > \tau \geq \sigma_{r^*+1}^2,$$

and in this case when  $\text{r}(\bar{R}^*) = r^*$ , we have shown that  $\lim_{\substack{(W_1)_{ij} \rightarrow \infty \\ (W_2)_{ij} \rightarrow 1}} \hat{R}_2 = \begin{pmatrix} R_{12} \\ H_{r^*} \begin{pmatrix} R_{22} \\ 0 \end{pmatrix} \end{pmatrix}$ .

Thus, together with (3.19), this implies

$$\begin{aligned} \lim_{\substack{(W_1)_{ij} \rightarrow \infty \\ (W_2)_{ij} \rightarrow 1}} (\hat{X}_1 \hat{X}_2) &= Q \left( \lim_{\substack{(W_1)_{ij} \rightarrow \infty \\ (W_2)_{ij} \rightarrow 1}} (\hat{R}_1 \hat{R}_2) \right) = Q \begin{pmatrix} R_{12} \\ R_1 \quad H_{r^*} \begin{pmatrix} R_{22} \\ 0 \end{pmatrix} \end{pmatrix} \\ &= (A_1 \quad Q_1 R_{12} + H_{r^*} \left( Q_2 \begin{pmatrix} R_{22} \\ 0 \end{pmatrix} \right)), \end{aligned}$$

which is the same as

$$(A_1 \quad P_{A_1}(A_2) + H_{r^*} (P_{A_1}^\perp(A_2))).$$

This completes the proof.  $\square$

**4. Numerical algorithm.** In this section we propose a numerical algorithm to solve a special case of (1.7), which, in general, does not have a closed form solution [7, 23]. Note (1.7) can be written as

$$\min_{\substack{\hat{X}_1, \hat{X}_2 \\ \text{r}(\hat{X}_1 \hat{X}_2) \leq r}} \left( \|(A_1 - \hat{X}_1) \odot W_1\|_F^2 + \|(A_2 - \hat{X}_2) \odot W_2\|_F^2 \right).$$

We assume that  $\text{r}(\hat{X}_1) = k$ . It can be verified that any  $\hat{X}_2$  such that  $\text{r}(\hat{X}_1 - \hat{X}_2) \leq r$  can be given in the form

$$\hat{X}_2 = \hat{X}_1 C + B D,$$

for some arbitrary matrices  $B \in \mathbb{R}^{m \times (r-k)}$ ,  $D \in \mathbb{R}^{(r-k) \times (n-k)}$ , and  $C \in \mathbb{R}^{k \times (n-k)}$ . Here we will focus on a special case when  $W_2 = \mathbb{1}$  in solving:

$$(4.1) \quad \min_{\hat{X}_1, C, B, D} \left( \|(A_1 - \hat{X}_1) \odot W_1\|_F^2 + \|A_2 - \hat{X}_1 C - B D\|_F^2 \right).$$

This will serve two purposes for us: One is to verify the rate given by Theorem 2.1 numerically and to gain some insight on the sharpness of the rate ( $O(\frac{1}{\lambda})$ , as  $\lambda \rightarrow \infty$ ); the other one is to demonstrate a numerical procedure based on alternating direction method in solving the weighted low-rank approximation problem that also allows detailed convergence analysis which is usually hard to obtain in other algorithms proposed in the literature [7, 23].

If  $k = 0$ , then (4.1) is an unweighted rank  $r$  factorization of  $A_2$  and is known as alternating least squares problem [13, 14, 15]. Denote  $F(\hat{X}_1, C, B, D) = \|(A_1 - \hat{X}_1) \odot W_1\|_F^2 + \|A_2 - \hat{X}_1 C - B D\|_F^2$  as the objective function. The above problem can be numerically solved by using an alternating strategy [4, 17] of minimizing the function with respect to each component iteratively:

$$(4.2) \quad \begin{cases} (\hat{X}_1)_{p+1} = \arg \min_{\hat{X}_1} F(\hat{X}_1, C_p, B_p, D_p), \\ C_{p+1} = \arg \min_C F((\hat{X}_1)_{p+1}, C, B_p, D_p), \\ B_{p+1} = \arg \min_B F((\hat{X}_1)_{p+1}, C_{p+1}, B, D_p), \\ \text{and, } D_{p+1} = \arg \min_D F((\hat{X}_1)_{p+1}, C_{p+1}, B_{p+1}, D). \end{cases}$$

Note that each of the minimizing problem for  $\hat{X}_1, C, B$ , and  $D$  can be solved explicitly by looking at the partial derivatives of  $F(\hat{X}_1, C, B, D)$ . But finding an update rule for  $\hat{X}_1$  turns out to be more involved than the other three variables. We update  $\hat{X}_1$  element wise along each row. Therefore we will use the notation  $\hat{X}_1(i, :)$  to denote the  $i$ -th row of the matrix  $\hat{X}_1$ . We set  $\frac{\partial}{\partial \hat{X}_1} F(\hat{X}_1, C_p, B_p, D_p)|_{\hat{X}_1 = (\hat{X}_1)_{p+1}} = 0$  and obtain

$$(4.3) \quad -(A_1 - (\hat{X}_1)_{p+1}) \odot W_1 \odot W_1 - (A_2 - (\hat{X}_1)_{p+1} C_p - B_p D_p) C_p^T = 0.$$

Solving the above expression for  $\hat{X}_1$  sequentially along each row gives

$$(\hat{X}_1(i, :))_{p+1} = (E(i, :))_p (\text{diag}(W_1^2(i, 1) \ W_1^2(i, 2) \cdots W_1^2(i, k)) + C_p C_p^T)^{-1},$$

where  $E_p = A_1 \odot W_1 \odot W_1 + (A_2 - B_p D_p) C_p^T$ . Similarly,  $C_{p+1}$  satisfies

$$\frac{\partial}{\partial C} F(\hat{X}_1, C, B_p, D_p)|_{C=C_{p+1}} = 0,$$

which implies

$$(4.4) \quad -(\hat{X}_1)_{p+1}^T (A_2 - (\hat{X}_1)_{p+1} C_{p+1} - B_p D_p) = 0,$$

and consequently can be solved as long as  $(\hat{X}_1)_{p+1}$  is of full rank. Therefore solving for  $C_{p+1}$  gives

$$C_{p+1} = ((\hat{X}_1)_{p+1}^T (\hat{X}_1)_{p+1})^{-1} ((\hat{X}_1)_{p+1}^T A_2 - (\hat{X}_1)_{p+1}^T B_p D_p).$$

Next we find  $B_{p+1}$  satisfies

$$(4.5) \quad -A_2 D_p^T + (\hat{X}_1)_{p+1} C_{p+1} D_p^T + B_{p+1} D_p D_p^T = 0.$$

Solving (4.5) for  $B_{p+1}$  obtains (assuming  $D_p$  is of full rank)

$$B_{p+1} = (A_2 D_p^T - (\hat{X}_1)_{p+1} C_{p+1} D_p^T) (D_p D_p^T)^{-1}.$$

Finally,  $D_{p+1}$  satisfies

$$(4.6) \quad -B_{p+1}^T A_2 + B_{p+1}^T (\hat{X}_1)_{p+1} C_{p+1} + B_{p+1}^T B_{p+1} D_{p+1} = 0,$$

and we can write (assuming  $B_{p+1}$  is of full rank)

$$D_{p+1} = (B_{p+1}^T B_{p+1})^{-1} (B_{p+1}^T A_2 - B_{p+1}^T (\hat{X}_1)_{p+1} C_{p+1}).$$

ALGORITHM 4.1.

1. *Inputs:*  $X \in \mathbb{R}^{m \times n}$  (the given matrix);  $W = (W_1 \ W_2) \in \mathbb{R}^{m \times n}$ ,  $W_2 = \mathbb{1} \in \mathbb{R}^{m \times (n-k)}$  (the weight); threshold  $\epsilon > 0$ .
2. *Initialization:*  $(\hat{X}_1)_0, C_0, B_0, D_0$ .
3. *While not converged do*  
 $E_p = A_1 \odot W_1 \odot W_1 + (A_2 - B_p D_p) C_p^T$   
 $(\hat{X}_1(i, :))_{p+1} = (E(i, :))_p (\text{diag}(W_1^2(i, 1) \ W_1^2(i, 2) \cdots W_1^2(i, k)) + C_p C_p^T)^{-1}$   
 $C_{p+1} = ((\hat{X}_1)_{p+1}^T (\hat{X}_1)_{p+1})^{-1} ((\hat{X}_1)_{p+1}^T A_2 - (\hat{X}_1)_{p+1}^T B_p D_p)$   
 $B_{p+1} = (A_2 D_p^T - (\hat{X}_1)_{p+1} C_{p+1} D_p^T) (D_p D_p^T)^{-1}$   
 $D_{p+1} = (B_{p+1}^T B_{p+1})^{-1} (B_{p+1}^T A_2 - B_{p+1}^T (\hat{X}_1)_{p+1} C_{p+1})$   
 $p = p + 1$
4. *Output:*  $(\hat{X}_1)_{p+1}, (\hat{X}_1)_{p+1} C_{p+1} + B_{p+1} D_{p+1}$ .

**4.1. Convergence analysis.** Next we will discuss the convergence of our numerical algorithm. The following equality will be very helpful.

THEOREM 4.2. *For a fixed  $(W_1)_{ij} > 0$ , and  $p = 1, 2, \dots$ , let  $m_p = F((\hat{X}_1)_p, C_p, B_p, D_p)$ .*

*Then,*

$$(4.7) \quad \begin{aligned} m_p - m_{p+1} = & \|((\hat{X}_1)_p - (\hat{X}_1)_{p+1}) \odot W_1\|_F^2 + \|((\hat{X}_1)_p - (\hat{X}_1)_{p+1}) C_p\|_F^2 \\ & + \|(\hat{X}_1)_{p+1} (C_p - C_{p+1})\|_F^2 + \|(B_p - B_{p+1}) D_p\|_F^2 + \|B_{p+1} (D_p - D_{p+1})\|_F^2. \end{aligned}$$

*Proof.* Denote

$$(4.8) \quad \begin{cases} m_p - F((\hat{X}_1)_{p+1}, C_p, B_p, D_p) = d_1, \\ F((\hat{X}_1)_{p+1}, C_p, B_p, D_p) - F((\hat{X}_1)_{p+1}, C_{p+1}, B_p, D_p) = d_2, \\ F((\hat{X}_1)_{p+1}, C_{p+1}, B_p, D_p) - F((\hat{X}_1)_{p+1}, C_{p+1}, B_{p+1}, D_p) = d_3, \\ \text{and, } F((\hat{X}_1)_{p+1}, C_{p+1}, B_{p+1}, D_p) - m_{p+1} = d_4. \end{cases}$$

Therefore,

$$(4.9) \quad \begin{aligned} d_1 = & \|(A_1 - (\hat{X}_1)_p) \odot W_1\|_F^2 + \|A_2 - (\hat{X}_1)_p C_p - B_p D_p\|_F^2 - \|(A_1 - (\hat{X}_1)_{p+1}) \odot W_1\|_F^2 \\ & - \|A_2 - (\hat{X}_1)_{p+1} C_p - B_p D_p\|_F^2 \\ = & \|(\hat{X}_1)_p \odot W_1\|_F^2 - \|(\hat{X}_1)_{p+1} \odot W_1\|_F^2 + \|(\hat{X}_1)_p C_p\|_F^2 - \|(\hat{X}_1)_p C_{p+1}\|_F^2 \\ & + 2\langle A_1 \odot W_1 \odot W_1, (\hat{X}_1)_{p+1} - (\hat{X}_1)_p \rangle - 2\langle ((\hat{X}_1)_p - (\hat{X}_1)_{p+1}) C_p, A_2 - B_p D_p \rangle. \end{aligned}$$

Note that,

$$(((\hat{X}_1)_{p+1} - A_1) \odot W_1 \odot W_1)((\hat{X}_1)_p - (\hat{X}_1)_{p+1})^T = (A_2 - (\hat{X}_1)_{p+1}C_p - B_pD_p)C_p^T((\hat{X}_1)_p - (\hat{X}_1)_{p+1})^T,$$

as  $(\hat{X}_1)_{p+1}$  satisfies (4.3). This, together with (4.9), will lead us to

$$(4.10) \quad d_1 = \|((\hat{X}_1)_p - (\hat{X}_1)_{p+1}) \odot W_1\|_F^2 + \|((\hat{X}_1)_p - (\hat{X}_1)_{p+1})C_p\|_F^2.$$

Similarly we find

$$(4.11) \quad \begin{cases} d_2 = \|(\hat{X}_1)_{p+1}(C_p - C_{p+1})\|_F^2, \\ d_3 = \|(B_p - B_{p+1})D_p\|_F^2, \\ d_4 = \|B_{p+1}(D_p - D_{p+1})\|_F^2. \end{cases}$$

Combining them together we have the desired result.  $\square$

Theorem 4.2 implies a lot of interesting convergence properties of the algorithm. For example, we have the following estimates.

**COROLLARY 4.3.** *We have*

- (i)  $m_p - m_{p+1} \geq \frac{1}{2}\|B_{p+1}D_{p+1} - B_pD_p\|_F^2$  for all  $p$ .
- (ii)  $m_p - m_{p+1} \geq \|((\hat{X}_1)_p - (\hat{X}_1)_{p+1}) \odot W_1\|_F^2$  for all  $p$ .

*Proof.* (i). From (4.7) we can write, for all  $p$ ,

$$\begin{aligned} m_p - m_{p+1} &\geq \|B_{p+1}(D_p - D_{p+1})\|_F^2 + \|(B_p - B_{p+1})D_p\|_F^2 \\ &= \frac{1}{2}(\|B_{p+1}D_{p+1} - B_pD_p\|_F^2 + \|2B_{p+1}D_p - B_{p+1}D_{p+1} - B_pD_p\|_F^2), \end{aligned}$$

by parallelogram identity. Therefore,

$$m_p - m_{p+1} \geq \frac{1}{2}\|B_{p+1}D_{p+1} - B_pD_p\|_F^2.$$

This completes the proof.

(ii). This follows immediately from (4.7).  $\square$

We now can state some convergence results as a consequence of Theorem 4.2 and Corollary 4.3.

**THEOREM 4.4.**

- (i) *We have the following:  $\sum_{p=1}^{\infty} \|B_{p+1}D_{p+1} - B_pD_p\|_F^2 < \infty$ , and*

$$\sum_{p=1}^{\infty} \left( \|((\hat{X}_1)_p - (\hat{X}_1)_{p+1}) \odot W_1\| \right) < \infty.$$

- (ii) *If  $\sum_{p=1}^{\infty} \sqrt{m_p - m_{p+1}} < +\infty$ , then  $\lim_{p \rightarrow \infty} B_pD_p$  and  $\lim_{p \rightarrow \infty} (\hat{X}_1)_p$  and exist. Furthermore if we write  $L^* := \lim_{p \rightarrow \infty} B_pD_p$  then  $\lim_{p \rightarrow \infty} B_{p+1}D_p = L^*$  for all  $p$ .*

*Proof.* (i). From Corollary 4.3 we can write, for  $N > 0$ ,

$$2(m_1 - m_{N+1}) \geq \sum_{p=1}^N (\|B_{p+1}D_{p+1} - B_pD_p\|_F^2),$$

$$\text{and } m_1 - m_{N+1} \geq \sum_{p=1}^{\infty} \left( \|((\hat{X}_1)_p - (\hat{X}_1)_{p+1}) \odot W_1\| \right) \geq \lambda^2 \sum_{p=1}^N \|(\hat{X}_1)_p - (\hat{X}_1)_{p+1}\|_F^2.$$

Recall,  $\lambda = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq k}} (W_1)_{ij}$ . Also note that,  $\{m_p\}_{p=1}^\infty$  is a decreasing non-negative

sequence. Hence the results follows.

(ii). Again using Corollary 4.3 we can write, for  $N > 0$ ,

$$\sum_{p=1}^N \sqrt{m_p - m_{p+1}} \geq \frac{1}{\sqrt{2}} \sum_{p=1}^N (\|B_{p+1}D_{p+1} - B_pD_p\|_F),$$

which implies  $\sum_{p=1}^\infty (B_{p+1}D_{p+1} - B_pD_p)$  is convergent if  $\sum_{p=1}^\infty \sqrt{m_p - m_{p+1}} < +\infty$ .

Therefore,  $\lim_{N \rightarrow \infty} B_N D_N$  exists. Similarly, we conclude  $\lim_{p \rightarrow \infty} (\hat{X}_1)_p$  exists.

Further,  $\lim_{p \rightarrow \infty} \|B_{p+1}D_{p+1} - B_pD_p\|_F^2 = 0$ , since  $\{m_p\}_{p=1}^\infty$  converges. Therefore  $\lim_{p \rightarrow \infty} B_{p+1}D_p$  exists and is equal to  $\lim_{p \rightarrow \infty} B_pD_p = L^*$ . This completes the proof.  $\square$

From Theorem 4.4, we can only prove the convergence of the sequence  $\{B_pD_p\}$  but not of  $\{B_p\}$  and  $\{D_p\}$  separately. We next establish the convergence of  $\{B_p\}$  and  $\{D_p\}$  with stronger assumption. Consider the situation when

$$(4.12) \quad \sum_{p=1}^\infty \sqrt{m_p - m_{p+1}} < +\infty.$$

THEOREM 4.5. Assume (4.12) holds.

- (i) If  $B_p$  is of full rank and  $B_p^T B_p \geq \gamma I_{r-k}$  for large  $p$  and some  $\gamma > 0$  then  $\lim_{p \rightarrow \infty} D_p$  exists.
- (ii) If  $D_p$  is of full rank and  $D_p D_p^T \geq \delta I_{r-k}$  for large  $p$  and some  $\delta > 0$  then  $\lim_{p \rightarrow \infty} B_p$  exists.
- (iii) If  $X_1^* := \lim_{p \rightarrow \infty} (\hat{X}_1)_p$  is of full rank, then  $C^* := \lim_{p \rightarrow \infty} C_p$  exists. Furthermore, if we write  $L^* = B^* D^*$ , for  $B^* \in \mathbb{R}^{m \times (r-k)}$ ,  $D^* \in \mathbb{R}^{(r-k) \times (n-k)}$ , then  $(X_1^*, C^*, B^*, D^*)$  will be a stationary point of  $F$ .

Proof. (i). Using (4.7) we have, for  $N > 0$ ,

$$\begin{aligned} \sum_{p=1}^N \sqrt{m_p - m_{p+1}} &\geq \sum_{p=1}^N \|B_{p+1}(D_p - D_{p+1})\|_F \\ &= \sum_{p=1}^N \sqrt{\text{tr}[(D_p - D_{p+1})^T B_{p+1}^T B_{p+1} (D_p - D_{p+1})]}, \end{aligned}$$

where  $\text{tr}(X)$  denotes the trace of the matrix  $X$ . Note that,  $B_p^T B_p \geq \gamma I_{r-k}$ , and we obtain

$$\sum_{p=1}^N \sqrt{m_p - m_{p+1}} \geq \sqrt{\gamma} \sum_{p=1}^N \|D_p - D_{p+1}\|_F,$$

which implies  $\sum_{p=1}^\infty (D_p - D_{p+1})$  is convergent if (4.12) holds. Therefore  $\lim_{N \rightarrow \infty} D_N$  exists. Similarly we can prove (ii).

(iii). Note that, from (4.7) we have, for  $N > 0$ ,

$$\begin{aligned} \sum_{p=1}^N \sqrt{m_p - m_{p+1}} &\geq \sum_{p=1}^N \|(\hat{X}_1)_{p+1}(C_p - C_{p+1})\|_F \\ &= \sum_{p=1}^N \sqrt{\text{tr}[(C_p - C_{p+1})^T (\hat{X}_1)_{p+1}^T (\hat{X}_1)_{p+1} (C_p - C_{p+1})]}. \end{aligned}$$

If  $X_1^* := \lim_{p \rightarrow \infty} (\hat{X}_1)_p$  is of full rank, it follows that, for large  $p$ ,  $(\hat{X}_1)_{p+1}^T (\hat{X}_1)_{p+1} \geq \eta I_k$ , for some  $\eta > 0$ . Therefore, we have

$$\sum_{p=1}^N \sqrt{m_p - m_{p+1}} \geq \sqrt{\eta} \sum_{p=1}^N \|C_p - C_{p+1}\|_F.$$

Following the same argument as in the previous proof we can conclude  $\lim_{p \rightarrow \infty} C_p = C^*$  exists if (4.12) holds. Recall from (4.3-4.6), we have,

$$\left\{ \begin{array}{l} ((\hat{X}_1)_{p+1} - A_1) \odot W_1 \odot W_1 - (A_2 - (\hat{X}_1)_{p+1} C_p - B_p D_p) C_p^T = 0, \\ (\hat{X}_1)_{p+1}^T (A_2 - (\hat{X}_1)_{p+1} C_{p+1} - B_p D_p) = 0, \\ (A_2 - (\hat{X}_1)_{p+1} C_{p+1} - B_{p+1} D_p) D_p^T = 0, \\ B_{p+1}^T (A_2 - (\hat{X}_1)_{p+1} C_{p+1} - B_{p+1} D_{p+1}) = 0. \end{array} \right.$$

Taking limit  $p \rightarrow \infty$  in above we have

$$\left\{ \begin{array}{l} \frac{\partial}{\partial X_1} F(X_1^*, C^*, B^*, D^*) = (X_1^* - A_1) \odot W_1 \odot W_1 + (B^* D^* + X_1^* C^* - A_2) C^{*T} = 0, \\ \frac{\partial}{\partial C} F(X_1^*, C^*, B^*, D^*) = X_1^{*T} (A_2 - X_1^* C^* - B^* D^*) = 0, \\ \frac{\partial}{\partial B} F(X_1^*, C^*, B^*, D^*) = (A_2 - X_1^* C^* - B^* D^*) D^{*T} = 0, \\ \frac{\partial}{\partial D} F(X_1^*, C^*, B^*, D^*) = B^{*T} (A_2 - X_1^* C^* - B^* D^*) = 0. \end{array} \right.$$

Therefore  $(X_1^*, C^*, B^*, D^*)$  is a stationary point of  $F$ . This completes the proof.  $\square$

**5. Numerical results.** In this section we will demonstrate numerical results of our weighted rank constrained algorithm and show the convergence to the solution given by Golub, Hoffman and Stewart when  $\lambda \rightarrow \infty$  as predicted by our theorems in Section 2. All experiments were performed on a computer with 3.1 GHz Intel Core i7-4770S processor and 8GB memory.

**5.1. Experimental setup.** We performed our experiments on three full rank synthetic matrices  $X$  of size  $300 \times 300$ ,  $500 \times 500$  and  $700 \times 700$  respectively. We constructed  $X$  as low rank matrix plus Gaussian noise such that  $X = X_0 + \alpha * E_0$ , where  $X_0$  is the low-rank matrix,  $E_0$  is the noise matrix, and  $\alpha$  controls the noise level. We generate  $X_0$  as a product of two independent full-rank matrices of size  $m \times r$  whose elements are independent and identically distributed (i.i.d.)  $\mathcal{N}(0, 1)$  random variables such that  $\text{r}(X_0) = r$ . We generate  $E_0$  as a noise matrix whose elements are i.i.d.  $\mathcal{N}(0, 1)$  random variables as well. In our experiments we choose  $\alpha = 0.2 \max_{i,j} (X_{ij})$ . The true rank of the test matrices are 10% of their original size but after adding noise they become full rank.



**5.2. Implementation details.** Let  $A_{WLR} = (\hat{X}_1^* \hat{X}_1^* C^* + B^* D^*)$  where  $(\hat{X}_1^*, C^*, B^*, D^*)$  be a solution to (4.1). We denote  $(A_{WLR})_p$  as our approximation to  $A_{WLR}$  at  $p$ th iteration. Recall that  $(A_{WLR})_p = ((\hat{X}_1)_p (\hat{X}_1)_p C_p + B_p D_p)$ . We denote  $\|(A_{WLR})_{p+1} - (A_{WLR})_p\|_F = \text{Error}_p$  and use  $\frac{\|(A_{WLR})_{p+1} - (A_{WLR})_p\|_F}{\|(A_{WLR})_p\|_F}$  as a measure of the relative error. For a threshold  $\epsilon > 0$  the stopping criteria of our algorithm at the  $p$ th iteration is  $\|(A_{WLR})_{p+1} - (A_{WLR})_p\|_F < \epsilon$  or  $\frac{\|(A_{WLR})_{p+1} - (A_{WLR})_p\|_F}{\|(A_{WLR})_p\|_F} < \epsilon$  or if it reaches the maximum iteration. The algorithm performs the best when we initialize  $\hat{X}_1$  and  $D$  as random normal matrices and  $B$  and  $C$  as zero matrices. Throughout this section we set  $r$  as the target low rank and  $k$  as the total number of columns we want to constrain in the observation matrix. The algorithm takes approximately 35.9973 seconds on an average to perform 2000 iterations on a  $300 \times 300$  matrix for fixed  $r, k$ , and  $\lambda$ .

**5.3. Experimental results on algorithm in section 4.1.** We first verify our implementation of the algorithm for computing  $A_{WLR}$  for fixed weights. We initialize our algorithm by random matrices. Throughout this subsection we set the target low-rank  $r$  as the true rank of the test matrix and  $k = 0.5r$ . To obtain the accurate result we run every experiment 25 times with random initialization and plot the average outcome in each case. A threshold equal to  $2.2204 \times 10^{-16}$  ("machine  $\epsilon$ ") is set for the experiments in this subsection. For Figure 5.1, we consider a nonuniform weight with entries in  $W_1$  randomly chosen from the interval  $[\lambda, \zeta]$ , where  $\min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq k}} (W_1)_{ij} = \lambda$  and  $\max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq k}} (W_1)_{ij} = \zeta$  in the first block  $W_1$  and  $W_2 = \mathbb{1}$  and plot iterations versus relative error. Relative error is plotted in logarithmic scale along Y-axis.

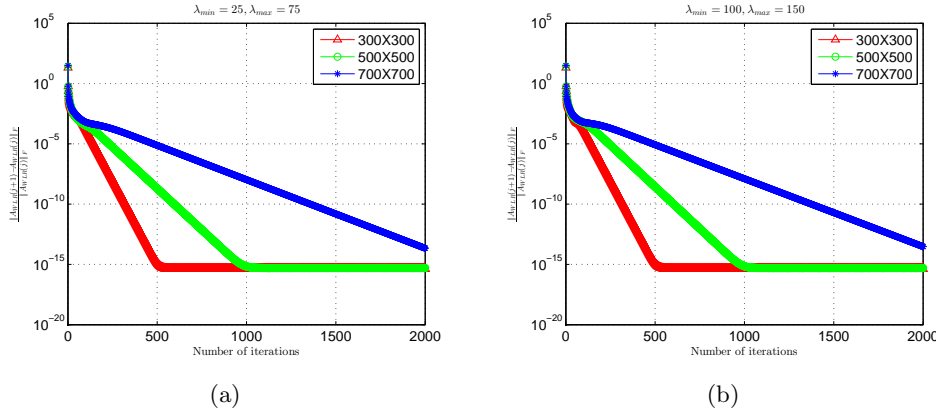


Fig. 5.1: Iterations vs Relative error: (a)  $\lambda = 25, \zeta = 75$ ; (b)  $\lambda = 100, \zeta = 150$ .

Next, we consider a uniform weight in the first block  $W_1$  and  $W_2 = \mathbb{1}$ . Recall that, in this case the solution to problem (1.7) can be given in closed form. by solving (1.6). That is, when  $W_1 = \lambda \mathbb{1}$ , the rank  $r$  solutions to (1.7) are  $X_{SVD} = [\frac{1}{\lambda} \tilde{X}_1 \tilde{X}_2]$ , where  $[\tilde{X}_1 \tilde{X}_2]$  is obtained in closed form using a SVD of  $[\lambda A_1 A_2]$ . In Figure 5.2, we plot iterations versus  $\frac{\|A_{WLR}(j) - X_{SVD}\|_F}{\|X_{SVD}\|_F}$  in logarithmic scale. From Figures 5.1 and 5.2 it is clear that the algorithm in Section 4.1 converges. Even for the bigger size matrices the iteration count is not very high to achieve the convergence.

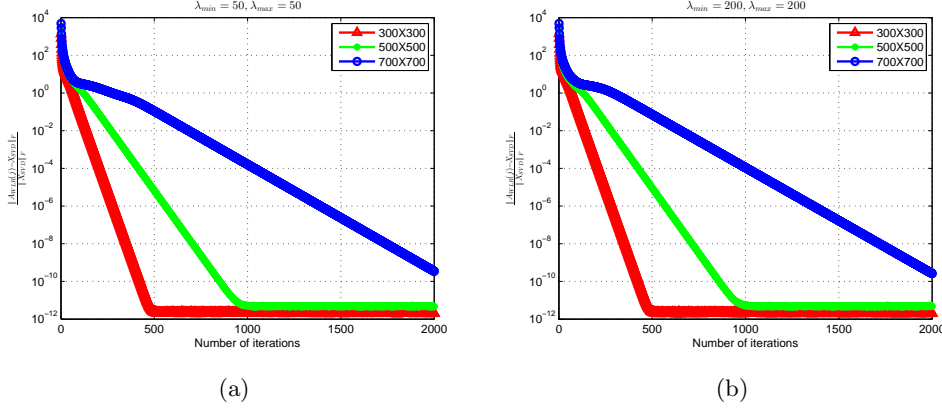


Fig. 5.2: Iterations vs  $\frac{\|A_{WLR}(j) - X_{SVD}\|_F}{\|X_{SVD}\|_F}$ : (a)  $\lambda = 50$ ; (b)  $\lambda = 200$ .

**5.4. Numerical results supporting Theorem 2.1.** We now demonstrate numerically the rate of convergence as stated in Theorem 2.1 when the block of weights in  $W_1$  goes to  $\infty$  and  $W_2 = 1$ . First we use an uniform weight  $W_1 = \lambda \mathbb{1}$  and  $W_2 = 1$ . The algorithm in Section 4 is used to compute  $A_{WLR}$  and SVD is used for calculating  $A_G$ , the solution to (1.4) when  $A = (A_1 \ A_2)$ . We plot  $\lambda$  vs.  $\lambda \|A_G - A_{WLR}\|_F$  where  $\lambda \|A_G - A_{WLR}\|_F$  is plotted in logarithmic scale along  $Y$ -axis. We run our algorithm 20 times with the same initialization and plot the average outcome. A threshold equal to  $10^{-7}$  is set for the experiments in this subsection. For Figure 5.3 we set  $\lambda = [1 : 50 : 1000]$ . The plots indicate for an uniform  $\lambda$  in  $W_1$  the con-

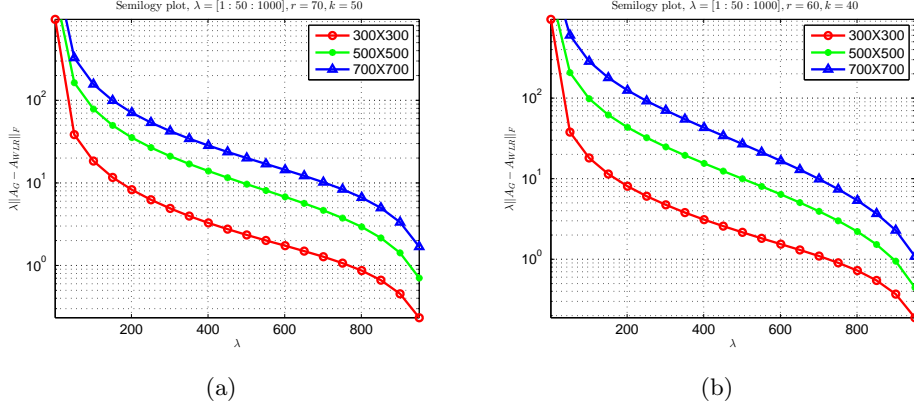


Fig. 5.3:  $\lambda$  vs.  $\lambda \|A_G - A_{WLR}\|_F$ : (a)  $(r, k) = (70, 50)$ , (b)  $(r, k) = (60, 40)$ .

vergence rate is at least  $O(\frac{1}{\lambda})$ ,  $\lambda \rightarrow \infty$ . Next we consider a nonuniform weight in the first block  $W_1$  and  $W_2 = 1$ . We consider  $\lambda = [2000 : 50 : 3000]$  such that  $(W_1)_{ij} \in [2000, 2020], [2050, 2070]$  and so on. For Figure 5.4,  $\lambda \|A_G - A_{WLR}\|_F$  is plotted in regular scale along  $Y$ -axis. The curves in figure 5.4 are not always strictly decreasing but it is encouraging to see that they stay bounded. Figures 5.3 and 5.4 pro-

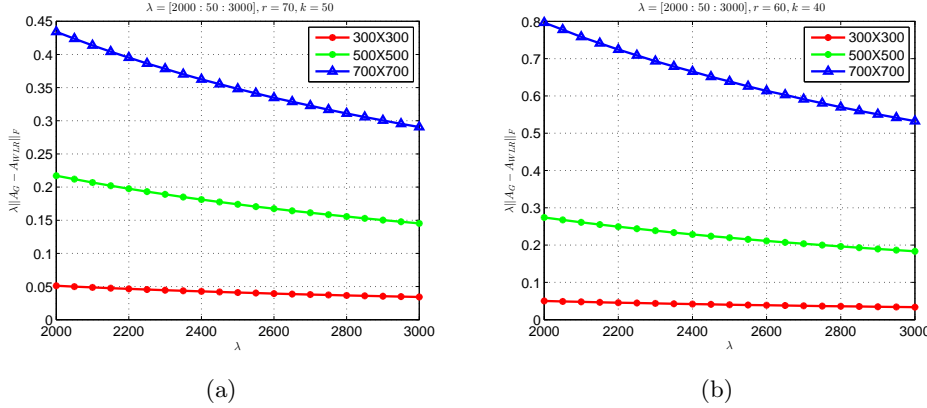


Fig. 5.4:  $\lambda$  vs.  $\lambda \|A_G - A_{WLR}\|_F$ : (a)  $(r, k) = (70, 50)$ , (b)  $(r, k) = (60, 40)$ .

vide numerical evidence in supporting Theorem 2.1. As established in Theorem 2.1 the above plots demonstrate the convergence rate is at least  $O(\frac{1}{\lambda})$ ,  $\lambda \rightarrow \infty$ .

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#### REFERENCES

- [1] G. H. GOLUB, A. HOFFMAN, AND G. W. STEWART, *A generalization of the Eckart-Young-Mirsky matrix approximation theorem*, Linear Algebra and its Applications, 88-89 (1987), pp. 317–327.
- [2] A. DUTTA, X. LI, B. GONG, AND M. SHAH, *A weighted singular value thresholding algorithm*, In Proceedings of Advances in Neural Information Processing Systems, 29 (2016), Submitted.
- [3] I. T. JOLLIFFEE, *Principal Component Analysis*, Second edition, Springer-Verlag, 2002, doi:10.1007/b98835.
- [4] Z. LIN, M. CHEN, AND Y. MA, *The augmented Lagrange multiplier method for exact recovery of corrupted low-rank matrices*, arXiv preprint arXiv1009.5055, 2010.
- [5] PER-ÅKE WEDIN, *Perturbation bounds in connection with singular value decomposition*, BIT Numerical Mathematics, 12-1(1972), pp. 99–111. doi:10.1007/BF01932678.
- [6] C. ECKART AND G. YOUNG, *The approximation of one matrix by another of lower rank*, Psychometrika, 1-3 (1936), pp. 211–218. doi:10.1007/BF02288367.
- [7] N. SREBRO AND T. JAAKKOLA, *Weighted low-rank approximations*, 20th International Conference on Machine Learning (2003), pp. 720–727.
- [8] G.W. STEWART, *A second order perturbation expansion for small singular values*, Linear Algebra and its Applications, 56 (1984), pp. 231–235, doi:10.1016/0024-3795(84)90128-9.
- [9] C. DAVIS AND W. KAHAN, *The rotation of eigenvectors by a perturbation III.*, SIAM Journal on Numerical Analysis, 7 (1970), pp. 1–46.
- [10] T. OKATANI AND K. DEGUCHI, *On the Wiberg algorithm for matrix factorization in the presence of missing components*, International Journal of Computer Vision, 72-3 (2007), pp. 329–337, doi: 10.1007/s11263-006-9785-5.
- [11] T. WIBERG, *Computation of principal components when data are missing*, In Proceedings of the Second Symposium of Computational Statistics (1976), pp. 229–336.

- [12] N. SREBRO, J. D. M. RENNIE, AND T. S. JAAKOLA, *Maximum-margin matrix factorization*, In Proc. of Advances in Neural Information Processing Systems, 18 (2005), pp. 1329–1336.
- [13] T. HASTIE, R. MAZUMDER, J. LEE, AND R. ZADEH, *Matrix completion and low-rank SVD via fast alternating least squares*, arXiv preprint arXiv:1410.2596, 2014.
- [14] M. UDELL, C. HORN, R. ZADEH, AND S. BOYD, *Generalized low-rank models*, arXiv preprint arXiv:1410.0342, 2014.
- [15] J. HANSOHN, *Some properties of the normed alternating least squares (ALS) algorithm*, Optimization, 19-5 (1988), pp. 683–691.
- [16] A. M. BUCHANAN AND A. W. FITZGIBBON, *Damped Newton algorithms for matrix factorization with missing data*, In Proceedings of the 2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition, 2 (2005), pp. 316–322, doi: 10.1109/CVPR.2005.118.
- [17] H. LIU, X. LI, AND X. ZHENG, *Solving non-negative matrix factorization by alternating least squares with a modified strategy*, Data Mining and Knowledge Discovery, 26-3 (2012), pp. 435–451, doi: 10.1007/s10618-012-0265-y.
- [18] I. MARKOVSKY, J. C. WILLEMS, B. DE MOOR, AND S. VAN HUFFEL, *Exact and approximate modeling of linear systems: a behavioral approach*, Number 11 in Monographs on Mathematical Modeling and Computation, SIAM, 2006.
- [19] I. MARKOVSKY, *Low-rank approximation: algorithms, implementation, applications*, Communications and Control Engineering. Springer, 2012.
- [20] S. VAN HUFFEL AND J. VANDEWALLE, *The total least squares problem: computational aspects and analysis*, Frontiers in Applied Mathematics 9, SIAM, Philadelphia, 1991.
- [21] K. USEVICH AND I. MARKOVSKY, *Variable projection methods for affinely structured low-rank approximation in weighted 2-norms*, Journal of Computational and Applied Mathematics 272 (2014), pp. 430–448.
- [22] G.W. STEWART, *On the asymptotic behavior of scaled singular value and QR decompositions*, Mathematics of Computation, 43-168 (1984), pp. 483–489.
- [23] J. H. MANTON, R. MEHONY, AND Y. HUA, *The geometry of weighted low-rank approximations*, IEEE Transactions on Signal Processing, 51-2 (2003), pp. 500–514.
- [24] W. S. LU, S. C. PEI, AND P. H. WANG, *Weighted low-rank approximation of general complex matrices and its application in the design of 2-D digital filters*, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 44-7 (1997), pp. 650–655, doi: 10.1109/81.596949.
- [25] D. SHPAK, *A weighted-least-squares matrix decomposition with application to the design of 2-D digital filters*, In Proceedings of IEEE 33rd Midwest Symposium on Circuits and Systems, (1990), pp. 1070–1073.
- [26] K. USEVICH AND I. MARKOVSKY, *Optimization on a Grassmann manifold with application to system identification*, Automatica, 50-6 (2014), pp. 1656–1662.